MH3100 Real Analysis I

Tutorial 6

Qikun Xiang

School of Physical and Mathematical Sciences,

Nanyang Technological University, Singapore

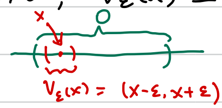
26 February, 2021

Recap : Week 6

Basic Topology of $\mathbb{R}$

(topologies are characterised by the open sets)

Open set : O is open $\Leftrightarrow \forall x \in O,\ \exists\, \varepsilon > 0,\ V_\varepsilon(x) \subseteq O$.

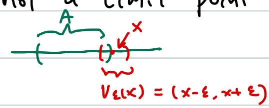


Intuitively speaking : "for every point $x \in O$, you can zoom in close enough to find a neighborhood of x entirely contained in O"

Limit point :

x is a **limit point** of $A \Leftrightarrow \forall \varepsilon > 0,\ \exists\, y \in V_\varepsilon(x) \cap A,\ y \neq x$

x is an **isolated point** of $A \Leftrightarrow$ x is not a limit point of A



Intuitively speaking : "no matter how close you zoom in, you can always find a point of A in the neighborhood of x other than x itself"

Closed set : C is closed

$\Leftrightarrow \forall x,\ (x \text{ is a limit point of } C \Rightarrow x \in C)$

"C contains all its limit points"

also : C is closed $\Leftrightarrow C^c$ is open (later)

Closure : the **closure** of A is $\bar{A} \Leftrightarrow \bar{A} = A \cup L$ where L is the set of limit points of A

Characterisation of open sets :

"arbitrary union of open sets is open" :

For any collection of subsets of $\mathbb{R}$ $(O_i)_{i\in I}$,

$(\forall i \in I,\ O_i \text{ is open}) \longrightarrow \bigcup_{i\in I} O_i$ is open.

"finite intersection of open sets is open" :

$\forall n \in \mathbb{N},\ (O_i \text{ is open for } 1 \le i \le n) \rightarrow \bigcap_{i=1}^{n} O_i$ is open.

Equivalent characterisation of limit points

x is a limit point of A $\Longleftrightarrow$ $\begin{cases} \exists (x_n) \subset A \\ x_n \neq x,\ \forall n \in \mathbb{N} \\ \lim_{n\to\infty} x_n = x \end{cases}$

In other words : " x is the limit of a sequence in A which does not include x "

Equivalent characterisation of closedness

C is closed $\Longleftrightarrow$ every Cauchy / convergent sequence in C has its limit in C

"arbitrary intersection of closed sets is closed" :

For any collection of subsets of $\mathbb{R}$ $(C_i)_{i\in I}$

$(\forall i \in I,\ C_i \text{ is closed}) \rightarrow \bigcap_{i\in I} C_i$ is closed

" finite union of closed sets is closed" :

$\forall n \in \mathbb{N},\ (C_i \text{ is closed for } 1 \le i \le n) \rightarrow \bigcup_{i=1}^{n} C_i$ is closed.

Equivalent characterisation of closure

$\bar{A}$ is the closure of A $\Leftrightarrow$ $\bar{A}$ is the intesection of sets $\{C \subseteq \mathbb{R} : A \subseteq C,\ C \text{ is closed}\}$

Check: • this intersection is closed because it is the intersection of (uncountably many) closed sets

• if C is closed and $C \supseteq A$, then $\bar{A} \subseteq C$.

Reminder: not open $\neq$ closed !

not closed $\neq$ open

Examples:

$(0,1]$ is not open since $V_\varepsilon(1) \not\subseteq (0,1]$ $\forall \varepsilon > 0$

$(0,1]$ is not closed since 0 is a limit point of $(0,1]$ yet $0 \notin (0,1]$.

A in Q3 is neither open nor closed.

1. Give an example of an infinite collection of nested open sets

$$O_0 \supseteq O_1 \supseteq O_2 \supseteq O_3 \supseteq \cdots,$$

whose intersection $\bigcap_{n=0}^{\infty} O_n$ is nonempty and closed.

Hint. See if any of these works:

(a) $O_n = (0, \frac{1}{n+1})$? (b) $O_n = (-\frac{1}{n+1}, \frac{1}{n+1})$? (c) $O_n = (-\frac{1}{n+1}, 1 + \frac{1}{n+1})$?

(a) $\bigcap_{n=0}^{\infty} (0, \frac{1}{n+1}) = \phi$. Proof: If $x \notin O_0 = (0,1)$ then $x \notin \bigcap_{n=0}^{\infty} (0, \frac{1}{n+1})$

If $x \in O_0 = (0,1)$, then there exists $N \in \mathbb{N}$ such that

$x > \frac{1}{N+1}$. Thus, $x \notin (0, \frac{1}{N+1})$ and $x \notin \bigcap_{n=0}^{\infty} (0, \frac{1}{n+1})$.

ϕ is empty, and in fact it is open (therefore it is not a correct example).

(b) $\bigcap_{n=0}^{\infty} (-\frac{1}{n+1}, \frac{1}{n+1}) = \{0\}$, which is non-empty and closed (also not open).

Proof: $0 \in (-\frac{1}{n+1}, \frac{1}{n+1})$ $\forall n \in \mathbb{N}$ $\Rightarrow 0 \in \bigcap_{n=0}^{\infty} (-\frac{1}{n+1}, \frac{1}{n+1})$

$\Rightarrow \{0\} \subseteq \bigcap_{n=0}^{\infty} (-\frac{1}{n+1}, \frac{1}{n+1})$.

Suppose $x \neq 0$. If $x \notin O_0 = (-1,1)$, then $x \notin \bigcap_{n=0}^{\infty} O_n$.

If $x \in (0,1)$, then there exists $N \in \mathbb{N}$ such that $x > \frac{1}{N+1}$. Thus, $x \notin (-\frac{1}{N+1}, \frac{1}{N+1})$ and $x \notin \bigcap_{n=0}^{\infty} (-\frac{1}{n+1}, \frac{1}{n+1})$.

If $x \in (-1,0)$, then there exists $N \in \mathbb{N}$ such that $x < -\frac{1}{N+1}$. Thus, $x \notin (-\frac{1}{N+1}, \frac{1}{N+1})$ and $x \notin \bigcap_{n=0}^{\infty} (-\frac{1}{n+1}, \frac{1}{n+1})$.

(c) $\bigcap_{n=0}^{\infty} \left(-\frac{1}{n+1}, 1+\frac{1}{n+1}\right) = [0,1]$ which is non-empty and closed (also not open). The proof is similar to example (b).

Infinite intersection of open sets may also be neither open nor closed.

For example, $\bigcap_{n=0}^{\infty} \left(0, 1+\frac{1}{n+1}\right) = (0,1]$.

2. Give an example of a collection of closed sets whose union is <u>not</u> closed.

<u>Hint.</u> Unfortunately, none of these works:

$$A_n = [0, \frac{1}{n+1}], \quad A_n = [-\frac{1}{n+1}, \frac{1}{n+1}], \quad A_n = [-\frac{1}{n+1}, 1 + \frac{1}{n+1}].$$

Try to modify it so that $A_n \subseteq A_{n+1}$.

In Q1, we were taking intersections and that's why we considered nested sequences ($O_{n+1} \subseteq O_n$).
In Q2, we are instead working with unions and therefore we should consider expanding sequences ($A_{n+1} \supseteq A_n$) to construct such an example.

The three given examples don't work because they are nested.
Hence, $\bigcup_{n=0}^{\infty} A_n = A_0$, which is closed.

Example that works: $A_n = [\frac{1}{n+1}, 1 - \frac{1}{n+1}]$. Notice the lower end points are decreasing, and the upper end points are increasing.

Claim: $\bigcup_{n=0}^{\infty} A_n = (0,1)$.

Proof: $\forall n \in \mathbb{N}$, $A_n \subseteq (0,1) \implies \bigcup_{n=0}^{\infty} A_n \subseteq (0,1)$

Let $x \in (0,1)$ be fixed. Then, there exists $N_1, N_2 \in \mathbb{N}$ such that $x > \frac{1}{N_1+1}$, $1-x > \frac{1}{N_2+1}$. Let $N = \max\{N_1, N_2\}$.

Then, $\frac{1}{N+1} \le \frac{1}{N_1+1} < x < 1 - \frac{1}{N_2+1} \le 1 - \frac{1}{N+1} \implies x \in [\frac{1}{N+1}, 1 - \frac{1}{N+1}] = A_N$
This shows that $(0,1) \subseteq \bigcup_{n=0}^{\infty} A_n$. Therefore, $\bigcup_{n=0}^{\infty} A_n = (0,1)$.

Similar to Q1, the union of infinitely many closed sets may be neither closed nor open. For example, consider

$$A_n = \left[\frac{1}{n+1}, 1\right], \quad \bigcup_{n=0}^{\infty} A_n = (0, 1].$$

3. Let $A = \left\{ (-1)^n \frac{n}{n+1} : n \in \mathbb{N} \right\}$. $= \left\{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \cdots \right\}$

(a) Show that 1 and -1 are two limit points of A. (We leave the justification that no other points are limit points of A to the Additional Exercise Question 2 below.)

(b) Is A an open set?

(c) Is A a closed set?

(d) Find the isolated points of A.

(e) Find $\overline{A}$, the closure of A.

(a) Let's prove that 1 is a limit point of A.

Let $\varepsilon > 0$ be fixed. Let $N \in \mathbb{N}$ be even such that $\varepsilon > \frac{1}{N+1}$.

Then, $1+\varepsilon > 1 - \frac{1}{N+1} > 1-\varepsilon$

$$\Rightarrow \begin{cases} 1 - \frac{1}{N+1} = (-1)^N \cdot \frac{N}{N+1} \in A, \\ 1 - \frac{1}{N+1} \in V_\varepsilon(1) = (1-\varepsilon, 1+\varepsilon), \\ 1 - \frac{1}{N+1} \neq 1. \end{cases}$$

Therefore, 1 is a limit point of A. $\square$

Similarly, -1 is also a limit point of A. The proof is analogous.

(b) No, A is not open. In fact, every non-empty countable set is not open.

Proof: Suppose A is non-empty and open. Let $x \in A$. Then, there exists $\varepsilon > 0$ such that $V_\varepsilon(x) \subseteq A$. But $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$ is a non-empty interval in $\mathbb{R}$, which contains uncountaby many numbers. Thus, A is uncountable. Now, take the contrapositive of the statement.

(c) No, A is not closed because 1 is a limit point of A yet $1 \notin A$.

(d) Since none of the points in A is a limit point of A, every point in A is an isolated point of A.

(e) $\overline{A} = A \cup \{-1, 1\}$.

4. Let $a \in A$. Prove that a is an isolated point of A if and only if there exists an ε-neighborhood $V_\varepsilon(a)$ such that $V_\varepsilon(a) \cap A = \{a\}$.

Proof:

Let $a \in A$

a is an isolated point of A

$\iff$ a is NOT a limit point of A

$\iff \neg\left[\forall \varepsilon > 0,\ (V_\varepsilon(a) \cap A) \setminus \{a\} \neq \phi\right]$

$\iff \exists\, \varepsilon > 0,\ (V_\varepsilon(a) \cap A) \setminus \{a\} = \phi$.

But
$$\left.\begin{array}{l} \exists\, \varepsilon > 0,\ (V_\varepsilon(a) \cap A) \setminus \{a\} = \phi \\ a \in V_\varepsilon(a) \\ a \in A \end{array}\right\} \iff \exists\, \varepsilon > 0,\ V_\varepsilon(a) \cap A = \{a\}.$$

Hence, a is an isolated point of $A \iff \exists\, \varepsilon > 0,\ V_\varepsilon(a) \cap A = \{a\}$.

$\square$.

We can strengthen the result a little bit and use it in Q5.

Now, we no longer assume that $a \in A$. From the derivation above,

a is an isolated point of A

$\iff \exists\, \varepsilon > 0,\ (V_\varepsilon(a) \cap A) \setminus \{a\} = \phi$.

But $\exists\, \varepsilon > 0,\ (V_\varepsilon(a) \cap A) \setminus \{a\} = \phi \iff \exists\, \varepsilon > 0,\ V_\varepsilon(a) \cap A \subseteq \{a\}$.

Hence, a is an isolated point of A if and only if $\exists\, \varepsilon > 0,\ V_\varepsilon(a) \cap A \subseteq \{a\}$.

5. Let A and B be any two sets of real numbers. Prove that if x is a limit point of $A \cup B$, then x is either a limit point of A or a limit point of B (maybe both).

Hint. We can prove this by contradiction.

Assume that x is neither a limit point of A nor a limit point of B. Then there exist δ_1 and $\delta_2 > 0$ such that the neighborhood $V_{\delta_1}(x) \cap A$ does not contain any element, except possibly x, and the neighborhood $V_{\delta_2}(x) \cap B$ does not contain any element, except possibly x.

What do we do with δ_1 and δ_2?

By the stronger version of Q4, there exist $\delta_1 > 0$, $\delta_2 > 0$, such that

$$V_{\delta_1}(x) \cap A \subseteq \{x\}, \quad V_{\delta_2}(x) \cap B \subseteq \{x\}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\begin{aligned} V_\delta(x) \cap (A \cup B) &= (V_\delta(x) \cap A) \cup (V_\delta(x) \cap B) \\ &\subseteq (V_{\delta_1}(x) \cap A) \cup (V_{\delta_2}(x) \cap B) \\ &\subseteq \{x\} \cup \{x\} = \{x\}. \end{aligned}$$

By the stronger version of Q4, x is an isolated point of $A \cup B$, contradicting the premise that x is a limit point of $A \cup B$. $\square$

6. Let A and B be any two sets of real numbers. Prove that the closure of $A \cup B$ is equal to the union of the closures of A and B.

Hint. To show "$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$": use Question 5.

To show "$\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$": it suffices to show $\overline{A \cup B} \supseteq \overline{A}$ and $\overline{A \cup B} \supseteq \overline{B}$. Why are these true?

Proof:

proof of "$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$":

Suppose that $x \in \overline{A \cup B}$

case 1 : $x \in A \cup B$

case 1.1 : $x \in A \Rightarrow x \in \overline{A} \Rightarrow x \in \overline{A} \cup \overline{B}$

case 1.2 : $x \in B \Rightarrow x \in \overline{B} \Rightarrow x \in \overline{A} \cup \overline{B}$.

Case 2: x is a limit point of $A \cup B$.

Then, by Q5, x is either a limit point of A or a limit point of B.

Case 2.1 : x is a limit point of $A \Rightarrow x \in \overline{A}$
$\Rightarrow x \in \overline{A} \cup \overline{B}$.

Case 2.2 : x is a limit point of $B \Rightarrow x \in \overline{B}$
$\Rightarrow x \in \overline{A} \cup \overline{B}$.

Therefore, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

proof of "$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$":

We use the fact that for two sets C, D,
$C \subseteq D \Rightarrow \overline{C} \subseteq \overline{D}$.

$$\left.\begin{array}{l} A \subseteq A \cup B \Rightarrow \overline{A} \subseteq \overline{A \cup B} \\ B \subseteq A \cup B \Rightarrow \overline{B} \subseteq \overline{A \cup B} \end{array}\right\} \Rightarrow \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

$\square$

Proof that $C \subseteq D \Rightarrow \overline{C} \subseteq \overline{D}$.

Suppose $C \subseteq D$ and $x \in \overline{C}$.

Case 1: $x \in C \Rightarrow x \in D \Rightarrow x \in \overline{D}$.

Case 2: x is a limit point of C

$$\Rightarrow \forall \varepsilon > 0, \ (V_\varepsilon(x) \cap C) \setminus \{x\} \neq \emptyset$$

$$\Rightarrow \forall \varepsilon > 0, \ (V_\varepsilon(x) \cap D) \setminus \{x\} \neq \emptyset$$

$\Rightarrow$ x is a limit point of D

$\Rightarrow x \in \overline{D}$.

Therefore, we conclude that $\overline{C} \subseteq \overline{D}$.

Alternatively, by the equivalent characterisation of closure,

$\overline{C} = \bigcap_{A \in \mathcal{A}_C} A$, where $\mathcal{A}_C = \{A \subseteq \mathbb{R} : A \text{ is closed}, A \supseteq C\}$

$\overline{D} = \bigcap_{A \in \mathcal{A}_D} A$, where $\mathcal{A}_D = \{A \subseteq \mathbb{R} : A \text{ is closed}, A \supseteq D\}$.

Since $C \subseteq D$, if $A \in \mathcal{A}_D$, then $A \in \mathcal{A}_C$. Hence, $\mathcal{A}_D \subseteq \mathcal{A}_C$.

$$\Rightarrow \overline{C} = \bigcap_{A \in \mathcal{A}_C} A = \underbrace{\left(\bigcap_{A \in \mathcal{A}_D} A\right)} \cap \left(\bigcap_{A \in \mathcal{A}_C \setminus \mathcal{A}_D} A\right)$$

$$= \overline{D} \cap \left(\bigcap_{A \in \mathcal{A}_C \setminus \mathcal{A}_D} A\right) \subseteq \overline{D}.$$

This is more intuitive. Since there are "more" closed sets containing C than closed sets containing D, $\overline{C}$ is the intersection of "more" sets compared to $\overline{D}$, hence $\overline{C} \subseteq \overline{D}$.