MH3100 Real Analysis I

Tutorial 7

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Recap : Week 7

Compactness

Definition :

K is compact $\iff$ every infinite sequence in K has a convergent subsequence with limit in K.

(actually, this is called sequential compactness)

An open cover of A is a collection of open sets $\{O_\lambda : \lambda \in \Lambda\}$ such that $\bigcup_{\lambda \in \Lambda} O_\lambda \supseteq A$.

Equivalent characterisations of compactness in $\mathbb{R}$: (or $\mathbb{R}^d$)

Heine-Borel Theorem
(a) K is compact ("every infinite sequence ...")
(b) K is closed and bounded
(c) every open cover of K has a finite subcover

Intuitively, a compact set behaves in many ways like a finite set.

e.g. A is compact $\Rightarrow$ $\min A$, $\max A$ exist

A is finite $\Rightarrow$ $\min A$, $\max A$ exist

From later

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous

A is compact $\Rightarrow$ $f(A)$ is compact

A is finite $\Rightarrow$ $f(A)$ is finite

1. Suppose that K is a compact set and F is a closed set. Prove that $K \cap F$ is a compact set.

<u>Hint.</u> Show $K \cap F$ is closed and bounded. Alternatively, show that $K \cap F$ satisfies the definition of compactness.

Approach 1: (by the Heine-Borel theorem)

Proof:

Since K is compact, we have by the Heine-Borel theorem that K is closed and bounded.

$\Rightarrow$ $K \cap F \subseteq K$ is also bounded.
$K \cap F$ is also closed because both K and F are closed.

Hence, by the Heine-Borel theorem, $K \cap F$ is compact. $\square$

Approach 2: (by definition)

Proof:

Suppose $(x_n)_{n \in \mathbb{N}} \subset K \cap F$ is an infinite sequence.

We have $(x_n)_{n \in \mathbb{N}} \subset K$, which implies that there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} x_{n_k} = x \in K$, by the compactness of K.

Notice that $(x_{n_k})_{k \in \mathbb{N}} \subset F$ is also a convergent sequence in F.
Hence, by the closedness of F, we have $x \in F$.

Therefore, $(x_{n_k})_{k \in \mathbb{N}} \subseteq (x_n)_{n \in \mathbb{N}} \subset K \cap F$ with

$$\lim_{k \to \infty} x_{n_k} = x \in K \cap F,$$

and thus $K \cap F$ is compact. $\square$

Approach 3: (open cover)

Goal: every open cover of $K \cap F$ admits a finite subcover.

Sketch of the proof:

1. Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of $K \cap F$.
2. F is closed $\Rightarrow$ F^c is open $\Rightarrow$ $O_\lambda \cup F^c$ is open $\forall \lambda \in \Lambda$.

(try to fill in the intermediate steps yourself)

3. $\{(O_\lambda \cup F^c) : \lambda \in \Lambda\}$ is an open cover of K.

   $\Rightarrow$ by the compactness of K, there exists a finite subset $\{(O_{\lambda_1} \cup F^c), (O_{\lambda_2} \cup F^c), \cdots, (O_{\lambda_n} \cup F^c)\} \subset \{O_\lambda : \lambda \in \Lambda\}$ such that $\bigcup_{i=1}^{n} (O_{\lambda_i} \cup F^c) \geq K$.

(try to fill in the intermediate steps yourself)

4. $\{O_{\lambda_n} : n = 1, \cdots, N\}$ is a finite subcover of $K \cap F$.

2. Decide whether the following sets are compact. For those that are not compact, give an example of a sequence contained in the given set that does not have a subsequence converging to a limit in the set.

(a) $\mathbb{Q}$; (b) $\mathbb{Q} \cap [0,1]$; (c) $\mathbb{R}$; (d) $\mathbb{R} \cap [0,1]$;

(e) $\{1,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ \ldots\}$; (f) $\{1,\ 1/2,\ 2/3,\ 3/4,\ 4/5,\ \ldots\}$.

(a) No. $a_n = n \quad \forall n \in \mathbb{N}$. Every subsequence is unbounded and is thus divergent.

(b) No. $a_n = \max\left\{\frac{m}{2^n} : m \in \mathbb{N},\ \frac{m}{2^n} < \frac{\sqrt{2}}{2}\right\} \quad \forall n \in \mathbb{N}$.
$(a_n)_{n \in \mathbb{N}} \subset \mathbb{Q} \cap [0,1]$.

$\lim_{n \to \infty} a_n = \frac{\sqrt{2}}{2} \notin \mathbb{Q} \cap [0,1]$ (Any subsequence converges to the same limit.)

(c) No. $a_n = n \quad \forall n \in \mathbb{N}$. Same as (a).

(d) Yes, because $\mathbb{R} \cap [0,1]$ is closed and bounded.

(e) No, $a_n = \frac{1}{n}$ $\lim_{n \to \infty} a_n = 0$ but $0 \notin \{1, \frac{1}{2}, \ldots\}$
(Any subsequence converges to the same limit.)

(f) Yes, because it is bounded and closed.

3. Prove that if nonempty K is compact, then both $\sup K$ and $\inf K$ exist and are elements of K.

Hint. Explain why $\sup K$ exists. To show $\sup K \in K$, show that there is some sequence $(x_n) \in K$ such that $(x_n) \to \sup K$. Proof of $\inf K \in K$ is similar.

Proof:

By the Heine-Borel theorem, K is closed and bounded.

$$\left.\begin{array}{l} K \text{ is bounded from above} \\ K \text{ is non-empty} \end{array}\right\} \Rightarrow \sup K \text{ exists.}$$

Let $\sup K = y \in \mathbb{R}$.

For every $n \in \mathbb{N}$, since $y - \frac{1}{n} < \sup K$, there exists $x_n \in K$ such that $y - \frac{1}{n} < x_n \leq \sup K = y$

Thus, $(x_n)_{n\in\mathbb{N}} \subset K$ is a sequence in K with $|x_n - y| < \frac{1}{n}$.

$\lim_{n\to\infty} |x_n - y| \leq \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} x_n = y = \sup K$.

and $\sup K \in K$ by the closedness of K.

(Recall that K is closed $\Leftrightarrow$ every Cauchy / convergent sequence in K has its limit in K).

The proof that $\inf K \in K$ is analogous. $\square$

4. Prove that the union of two compact sets is compact via the Heine-Borel theorem.

Hint. Show that $A \cup B$ is closed and bounded.

Proof: Suppose A and B are both compact.

By the Heine-Borel theorem:

A is compact $\Rightarrow$ A is closed and bounded

B is compact $\Rightarrow$ B is closed and bounded

Consequently, the closedness of $A \cup B$ follows from the closedness of A and B (because it is a finite union).

Regarding boundedness:

$\exists M_1 \in \mathbb{R} \quad |a| \leq M_1 \quad \forall a \in A$

$\exists M_2 \in \mathbb{R} \quad |b| \leq M_2 \quad \forall b \in B$

Take $M = \max\{M_1, M_2\}$

$\forall x \in A \cup B, \quad |x| \leq M \Rightarrow A \cup B$ is bounded.

Therefore, by the Heine-Borel theorem, $A \cup B$ is compact. $\square$

Proof by definition: Suppose A and B are both compact. Let $(x_n)_{n\in\mathbb{N}} \subset A \cup B$ be an infinite sequence. Then, one of $((x_n)_{n\in\mathbb{N}} \cap A)$ and $((x_n)_{n\in\mathbb{N}} \cap B)$ must be infinite. Assume without loss of generality that $((x_n)_{n\in\mathbb{N}} \cap A)$ is infinite. Then, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset A$, which, by the compactness of A, admits a further subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} x_{n_{k_j}} = x \in A$ and thus $(x_n)_{n\in\mathbb{N}}$ admits a subsequence with limit in $A \cup B$. Therefore, $A \cup B$ is compact. $\square$

5. Prove that the union of two compact sets is compact by using the property that every open cover for a compact set K has a finite subcover.

<u>Hint.</u> Union of two finite subcovers is still a finite subcover.

Proof: Suppose A and B are both compact.

Let $\{O_\lambda : \lambda \in \Lambda\}$ be an arbitrary open cover of $A \cup B$,

i.e. $\bigcup_{\lambda \in \Lambda} O_\lambda \supseteq A \cup B$.

$\Rightarrow \bigcup_{\lambda \in \Lambda} O_\lambda \supseteq A$ and $\bigcup_{\lambda \in \Lambda} O_\lambda \supseteq B$

$\Rightarrow \{O_\lambda : \lambda \in \Lambda\}$ is an open cover of A, and
$\{O_\lambda : \lambda \in \Lambda\}$ is an open cover of B.

Since A and B are compact, there exist $\Lambda_A \subset \Lambda$, $\Lambda_B \subset \Lambda$ with $|\Lambda_A| < \infty$, $|\Lambda_B| < \infty$, such that

$$\bigcup_{\lambda \in \Lambda_A} O_\lambda \supseteq A, \quad \bigcup_{\lambda \in \Lambda_B} O_\lambda \supseteq B,$$

i.e. $\{O_\lambda : \lambda \in \Lambda_A\}$ is a finite subcover of A, and
$\{O_\lambda : \lambda \in \Lambda_B\}$ is a finite subcover of B.

Let $\tilde{\Lambda} = \Lambda_A \cup \Lambda_B$. We have $|\tilde{\Lambda}| < \infty$, $\tilde{\Lambda} \subset \Lambda$.

$$\bigcup_{\lambda \in \tilde{\Lambda}} O_\lambda = \left(\bigcup_{\lambda \in \Lambda_A} O_\lambda\right) \cup \left(\bigcup_{\lambda \in \Lambda_B} O_\lambda\right) \supseteq A \cup B.$$ Then,

$\{O_\lambda : \lambda \in \tilde{\Lambda}\}$ is a finite subcover of $A \cup B$.

Therefore, $A \cup B$ is compact. $\square$

6. Give an example of a set X, and an open cover of X that does not have a finite subcover.

<u>Hint.</u> Consider either X unbounded or not closed.

(a) $\mathbb{Q}$; (b) $\mathbb{Q} \cap [0,1]$; (c) $\mathbb{R}$; (d) $\mathbb{R} \cap [0,1]$;

(e) $\{1,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ \ldots\}$; (f) $\{1,\ 1/2,\ 2/3,\ 3/4,\ 4/5,\ \ldots\}$.

(b) $a_n = \max\left\{ \frac{m}{2^{n+1}} : m \in \mathbb{N},\ \frac{m}{2^{n+1}} < \frac{\sqrt{2}}{2} \right\}$ $\forall n \in \mathbb{N}$

$b_n = \min\left\{ \frac{m}{2^{n+1}} : m \in \mathbb{N},\ \frac{m}{2^{n+1}} > \frac{\sqrt{2}}{2} \right\}$ $\forall n \in \mathbb{N}$

Let $O_n = (-\frac{1}{2}, a_n) \cup (b_n, \frac{3}{2})$ $\forall n \in \mathbb{N}$.

Then, $\bigcup_{n=1}^{\infty} O_n = (-\frac{1}{2}, \frac{3}{2}) \setminus \{\frac{\sqrt{2}}{2}\} \supseteq \mathbb{Q} \cap [0,1]$.

For any finite subset $\{O_{n_1}, O_{n_2}, \cdots, O_{n_m}\} \subset \{O_n : n \in \mathbb{N}\}$,

we have $\bigcup_{k=1}^{m} O_{n_k} = O_{n_m} = (-\frac{1}{2}, a_{n_m}) \cup (b_{n_m}, \frac{3}{2})$

We know $a_{n_m} < \frac{\sqrt{2}}{2} < b_{n_m}$, and thus

$\frac{a_{n_m} + b_{n_m}}{2} \in \mathbb{Q} \cap [0,1]$ but $\frac{a_{n_m} + b_{n_m}}{2} \notin \bigcup_{k=1}^{m} O_{n_k}$.

Hence, $\{O_n : n \in \mathbb{N}\}$ does not admit a finite subcover.

(c) Let $O_n = (-n, n)$ $\forall n \in \mathbb{N}$. Then, $\bigcup_{n=1}^{\infty} O_n = \mathbb{R}$.

For any subset $\{O_{n_1}, O_{n_2}, \cdots, O_{n_m}\} \subset \{O_n : n \in \mathbb{N}\}$,
we have $\bigcup_{k=1}^{m} O_{n_k} = O_{n_m} = (-n_m, n_m) \not\supseteq \mathbb{R}$.

Hence, $\{O_n : n \in \mathbb{N}\}$ does not admit a finite subcover.